The derivation of local volatility is outlined in many papers and textbooks (such as the one by Jim Gatheral [1]), but in the derivations many steps are left out. In this Note we provide two derivations of local volatility.

1. The derivation by Dupire [2] that uses the Fokker-Planck equation.
2. The derivation by Derman et al. [3] of local volatility as a conditional expectation.

We also present the derivation of local volatility from Black-Scholes implied volatility, outlined in [1]. We will derive the following three equations that involve local volatility \( \sigma = \sigma(S_t, t) \) or local variance \( \nu_L = \sigma^2 \).

1. The Dupire equation in its most general form (appears in [1] on page 9)

   \[
   \frac{\partial C}{\partial T} = \frac{1}{2} \sigma^2 K \frac{\partial^2 C}{\partial K^2} + (r_T - q_T) \left( C - K \frac{\partial C}{\partial K} \right) - r_T C. \tag{1}
   \]

2. The equation by Derman et al. [3] for local volatility as a conditional expected value (appears with \( q_T = 0 \) in [3])

   \[
   \frac{\partial C}{\partial T} = -K(r_T - q_T) \frac{\partial C}{\partial K} - q_T C + \frac{1}{2} K^2 E \left[ \sigma_T^2 | S_T = K \right] \frac{\partial^2 C}{\partial K^2}. \tag{2}
   \]

3. Local volatility as a function of Black-Scholes implied volatility, \( \Sigma = \Sigma(K, T) \) (appears in [1]) expressed here as the local variance \( \nu_L \)

   \[
   \nu_L = \frac{\partial w}{\partial T} \left[ 1 - \frac{y}{\nu} \frac{\partial w}{\partial y} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2} + \frac{1}{4} \left( -\frac{1}{4} - \frac{1}{\nu} + \frac{y^2}{\nu^2} \right) \left( \frac{\partial w}{\partial y} \right)^2 \right]. \tag{3}
   \]

   where \( w = \Sigma(K, T)^2 T \) is the Black-Scholes total implied variance and \( y = \ln \frac{K}{F_T} \) where \( F_T = \exp \left( \int_0^T \mu_t dt \right) \) is the forward price with \( \mu_t = r_t - q_t \) (risk free rate minus dividend yield). Alternatively, local volatility can also be expressed in terms of \( \Sigma \) as

   \[
   \frac{\Sigma^2 + 2 \Sigma T \left[ \frac{\partial \Sigma}{\partial T} + (r_T - q_T) K \frac{\partial \Sigma}{\partial K} \right]}{\left( 1 + K y \frac{\partial \Sigma}{\partial K} \right)^2} + \frac{K \Sigma T}{\frac{\partial \Sigma}{\partial K} - \frac{1}{4} K \Sigma T \left( \frac{\partial \Sigma}{\partial K} \right)^2 + K \frac{\partial^2 \Sigma}{\partial K^2}}. \]

Solving for the local variance in Equation (1), we obtain

\[
\sigma^2 = \sigma(K, T)^2 = \frac{\frac{\partial C}{\partial T} - (r_T - q_T) (C - K \frac{\partial C}{\partial K})}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}. \tag{4}
\]
If we set the risk-free rate \( r_T \) and the dividend yield \( q_T \) each equal to zero, Equations (1) and (2) can each be solved to yield the same equation involving local volatility, namely

\[
\sigma^2 = \sigma(K, T)^2 = -\frac{\partial C}{\partial T} \frac{1}{2K^2} \frac{\partial^2 C}{\partial K^2}.
\]  

(5)

The local volatility is then \( v_L = \sqrt{\sigma^2(K, T)} \). In this Note the derivation of these equations are all explained in detail.

1 Local Volatility Model for the Underlying

The underlying \( S_t \) follows the process

\[
dS_t = \mu S_t dt + \sigma(S_t, t) S_t dW_t
\]

(6)

\[
= (r_t - q_t) S_t dt + \sigma(S_t, t) S_t dW_t.
\]

We sometimes drop the subscript and write \( dS = \mu S dt + \sigma S dW \) where \( \sigma = \sigma(S_t, t) \). We need the following preliminaries:

- Discount factor \( P(t, T) = \exp \left(-\int_t^T r_s ds\right) \).
- Fokker-Planck equation. Denote by \( f(S_t, t) \) the probability density function of the underlying price \( S_t \) at time \( t \). Then \( f \) satisfies the equation

\[
\frac{\partial f}{\partial t} = -\frac{\partial}{\partial S} [\mu S f(S, t)] + \frac{1}{2} \frac{\partial^2}{\partial S^2} \left[ \sigma^2 S^2 f(S, t) \right].
\]

(7)

- Time-\( t \) price of European call with strike \( K \), denoted \( C = C(S_t, K) \)

\[
C = P(t, T) E \left[ (S_T - K)^+ \right] = P(t, T) E \left[ (S_T - K) 1_{(S_T > K)} \right] = P(t, T) \int_K^\infty (S_T - K) f(S, T) dS.
\]

(8)

where \( 1_{(S_T > K)} \) is the Heaviside function and where \( E[\cdot] = E[\cdot|\mathcal{F}_t] \). In the all the integrals in this Note, since the expectations are taken for the underlying price at \( t = T \) it is understood that \( S = S_T, f(S, T) = f(S_T, T) \) and \( dS = dS_T \). We sometimes omit the subscript for notational convenience.

2 Derivation of the General Dupire Equation (1)

2.1 Required Derivatives

We need the following derivatives of the call \( C(S_t, t) \).

2...
• First derivative with respect to strike

\[
\frac{\partial C}{\partial K} = P(t, T) \int_K^\infty \frac{\partial}{\partial K} (S_T - K) f(S, T) dS
\]

(9)

\[
= -P(t, T) \int_K^\infty f(S, T) dS.
\]

• Second derivative with respect to strike

\[
\frac{\partial^2 C}{\partial K^2} = -P(t, T) \left[ f(S, T) \right]_{S=\infty}^{S=K}
\]

(10)

\[
= P(t, T) f(K, T).
\]

We have assumed that \( \lim_{S \to \infty} f(S, T) = 0. \)

• First derivative with respect to maturity–use the chain rule

\[
\frac{\partial C}{\partial T} = \frac{\partial C}{\partial T} P(t, T) \times \int_K^\infty (S_T - K) f(S, T) dS +
\]

(11)

\[
P(t, T) \times \int_K^\infty (S_T - K) \frac{\partial}{\partial T} \left[ f(S, T) \right] dS.
\]

Note that \( \frac{\partial P}{\partial T} = -r_T P(t, T) \) so we can write (11)

\[
\frac{\partial C}{\partial T} = -r_T C + P(t, T) \int_K^\infty (S_T - K) \frac{\partial}{\partial T} \left[ f(S, T) \right] dS.
\]

(12)

2.2 Main Equation

In Equation (12) substitute the Fokker-Planck equation (7) for \( \frac{\partial f}{\partial t} \) at \( t = T \)

\[
\frac{\partial C}{\partial T} + r_T C = P(t, T) \int_K^\infty (S_T - K) \times
\]

\[
\left\{- \frac{\partial}{\partial S} [\mu_T S f(S, T)] + \frac{1}{2} \frac{\partial^2}{\partial S^2} \left[ \sigma^2 S^2 f(S, T) \right] \right\} dS.
\]

(13)

This is the main equation we need because it is from this equation that the Dupire local volatility is derived. In Equation (13) have two integrals to evaluate

\[
I_1 = \mu_T \int_K^\infty (S_T - K) \frac{\partial}{\partial S} [S f(S, T)] dS,
\]

(14)

\[
I_2 = \int_K^\infty (S_T - K) \frac{\partial^2}{\partial S^2} \left[ \sigma^2 S^2 f(S, T) \right] dS.
\]

Before evaluating these two integrals we need the following two identities.
2.3 Two Useful Identities

2.3.1 First Identity

From the call price Equation (8), we obtain
\[
\frac{C}{P(t,T)} = \int_{K}^{\infty} (S_T - K) f(S,T) dS = \int_{K}^{\infty} S_T f(S,T) dS - K \int_{K}^{\infty} f(S,T) dS.
\] (15)

From the expression for \( \frac{\partial C}{\partial K} \) in Equation (9) we obtain
\[
\int_{K}^{\infty} f(S,T) dS = -\frac{1}{P(t,T)} \frac{\partial C}{\partial K}.
\]

Substitute back into Equation (15) and re-arrange terms to obtain the first identity
\[
\int_{K}^{\infty} S_T f(S,T) dS = \frac{C}{P(t,T)} - \frac{K}{P(t,T)} \frac{\partial C}{\partial K}.
\] (16)

2.3.2 Second Identity

We use the expression for \( \frac{\partial^2 C}{\partial K^2} \) in Equation (10) to obtain the second identity
\[
f(K,T) = \frac{1}{P(t,T)} \frac{\partial^2 C}{\partial K^2}.
\] (17)

2.4 Evaluating the Integrals

We can now evaluate the integrals \( I_1 \) and \( I_2 \) defined in Equation (14).

2.4.1 First integral

Use integration by parts with \( u = S_T - K, u' = 1, v' = \frac{\partial}{\partial S} [S f(S,T)], v = S f(S,T) \)
\[
I_1 = [\mu_T (S_T - K) S_T f(S,T)]_{S=K}^{S=\infty} - \mu_T \int_{K}^{\infty} S f(S,T) dS
= [0 - 0] - \mu_T \int_{K}^{\infty} S f(S,T) dS.
\]

We have assumed \( \lim_{S \to \infty} (S - K) S f(S,T) = 0 \). Substitute the first identity (16) to obtain the first integral \( I_1 \)
\[
I_1 = \frac{-\mu_T C}{P(t,T)} + \frac{\mu_T K}{P(t,T)} \frac{\partial C}{\partial K}.
\] (18)
2.4.2 Second integral

Use integration by parts with \( u = ST - K, u' = 1, v' = \frac{\partial}{\partial S} \left[ \sigma^2 S^2 f(S, T) \right], v = \frac{\partial}{\partial S} \left[ \sigma^2 S^2 f(S, T) \right] \)

\[
I_2 = \left[ (ST - K) \frac{\partial}{\partial S} \left\{ \sigma^2 S^2 f(S, T) \right\} \right]_{S=K}^{S=\infty} - \int_K^\infty \frac{\partial}{\partial S} \left[ \sigma^2 S^2 f(S, T) \right] dS
\]

\[
= \left[ 0 - 0 \right] - \left[ \sigma^2 S^2 f(S, T) \right]_{S=K}^{S=\infty}
\]

\[
= \sigma^2 K^2 f(K, T)
\]

where \( \sigma^2 = \sigma(K, T)^2 \). We have assumed that \( \lim_{S \to \infty} \frac{\partial}{\partial S} \left\{ \sigma^2 S^2 f(S, T) \right\} = 0 \).

Substitute the second identity (17) for \( f(K, T) \) to obtain the second integral \( I_2 \)

\[
I_2 = \frac{\sigma^2 K^2}{P(t, T)} \frac{\partial^2 C}{\partial K^2}. \tag{19}
\]

2.5 Obtaining the Dupire Equation

We can now evaluate the main Equation (13) which we write as

\[
\frac{\partial C}{\partial T} + r_T C = P(t, T) \left[ -I_1 + \frac{1}{2} I_2 \right].
\]

Substitute for \( I_1 \) from Equation (18) and for \( I_2 \) from Equation (19)

\[
\frac{\partial C}{\partial T} + r_T C = \mu_T C - \mu_T K \frac{\partial C}{\partial K} + \frac{1}{2} \sigma^2 K^2 \frac{\partial^2 C}{\partial K^2}
\]

Substitute for \( \mu_T = r_T - q_T \) (risk free rate minus dividend yield) to obtain the Dupire equation (1)

\[
\frac{\partial C}{\partial T} = \frac{1}{2} \sigma^2 K^2 \frac{\partial^2 C}{\partial K^2} + (r_T - q_T) \left( C - K \frac{\partial C}{\partial K} \right) - r_T C.
\]

Solve for \( \sigma^2 = \sigma(K, T)^2 \) to obtain the Dupire local variance in its general form

\[
\sigma(K, T)^2 = \frac{\frac{\partial C}{\partial T} + q_T C + (r_T - q_K) K \frac{\partial C}{\partial K}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}
\]

Dupire [2] assumes zero interest rates and zero dividend yield. Hence \( r_T = q_T = 0 \) so that the underlying process is \( dS_t = \sigma(S_t, t) dW_t \). We obtain

\[
\sigma(K, T)^2 = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}.
\]

which is Equation (5).
3 Derivation of Local Volatility as an Expected Value, Equation (2)

We need the following preliminaries, all of which are easy to show

\[
\begin{align*}
\frac{\partial}{\partial S} (S-K)^+ &= 1_{(S>K)} \\
\frac{\partial}{\partial K} (S-K)^+ &= -1_{(S>K)} \\
\frac{\partial C}{\partial K} &= -P(t,T)E[1_{(S>K)}] \\
\frac{\partial^2 C}{\partial K^2} &= P(t,T)E[\delta(S-K)] \\
\end{align*}
\]

In the table, \(\delta(\cdot)\) denotes the Dirac delta function. Now define the function \(f(S_T, T)\) as

\[ f(S_T, T) = P(t, T)(S_T - K)^+ \]

Recall the process for \(S_t\) is given by Equation (6). By Itô’s Lemma, \(f\) follows the process

\[ df = \left[ \frac{\partial f}{\partial T} + \mu_T S_T \frac{\partial f}{\partial S_T} + \frac{1}{2} \sigma_T^2 S_T^2 \frac{\partial^2 f}{\partial S_T^2} \right] dT + \left[ \sigma_T S_T \frac{\partial f}{\partial S_T} \right] dW_T. \tag{20} \]

Now the partial derivatives are

\[
\begin{align*}
\frac{\partial f}{\partial T} &= -r_T P(t,T)(S_T - K)^+, \\
\frac{\partial f}{\partial S_T} &= P(t,T)1_{(S_T > K)}, \\
\frac{\partial^2 f}{\partial S_T^2} &= P(t,T)\delta(S_T - K). \\
\end{align*}
\]

Substitute them into Equation (20)

\[
\begin{align*}
df &= P(t,T) \times \\
&\quad \left[ -r_T(S_T - K)^+ + \mu_T S_T 1_{(S_T > K)} + \frac{1}{2} \sigma_T^2 S_T^2 \delta(S_T - K) \right] dT \\
&\quad + P(t,T) \left[ \sigma_T S_T 1_{(S_T > K)} \right] dW_T
\end{align*}
\]

Consider the first two terms of (21), which can be written as

\[
\begin{align*}
-r_T(S_T - K)^+ + \mu_T S_T 1_{(S_T > K)} &= -r_T(S_T - K)1_{(S_T > K)} + \mu_T S_T 1_{(S_T > K)} \\
&= r_T K 1_{(S_T > K)} - q_T S_T 1_{(S_T > K)}. \\
\end{align*}
\]

When we take the expected value of Equation (21), the stochastic term drops out since \(E[dW_T] = 0\). Hence we can write the expected value of (21) as

\[ dC = E[df] \]

\[ = P(t,T)E \left[ r_T K 1_{(S_T > K)} - q_T S_T 1_{(S_T > K)} + \frac{1}{2} \sigma_T^2 S_T^2 \delta(S_T - K) \right] dT \]
so that
\[ \frac{dC}{dT} = P(t, T)E \left[ r_T S_T 1_{(S_T > K)} - q_T S_T 1_{(S_T > K)} + \frac{1}{2} \sigma_T^2 S_T^2 \delta(S_T - K) \right] . \]  
(23)

Using the second line in Equation (8), we can write
\[ P(t, T)E \left[ S_T 1_{(S_T > K)} \right] = C + KP(t, T)E \left[ 1_{(S_T > K)} \right] \]
so Equation (23) becomes
\[ \frac{dC}{dT} = KP(t, T)E \left[ 1_{(S_T > K)} \right] \]
\[ + \frac{1}{2} P(t, T)E \left[ \sigma_T^2 S_T^2 \delta(S_T - K) \right] \]
\[ = -K(r_T - q_T) \frac{\partial C}{\partial K} - q_T C + \frac{1}{2} P(t, T)E \left[ \sigma_T^2 S_T^2 \delta(S_T - K) \right] \]
where we have substituted \(-\frac{\partial C}{\partial K}\) for \(P(t, T)E[1_{(S_T > K)}]\). The last term in the last line of Equation (24) can be written
\[ \frac{1}{2} P(t, T)E \left[ \sigma_T^2 S_T^2 \delta(S_T - K) \right] = \frac{1}{2} P(t, T)E \left[ \sigma_T^2 S_T^2 | S_T = K \right] E[\delta(S_T - K)] \]
\[ = \frac{1}{2} P(t, T)E \left[ \sigma_T^2 | S_T = K \right] K^2 E[\delta(S_T - K)] \]
\[ = \frac{1}{2} E \left[ \sigma_T^2 | S_T = K \right] K^2 \frac{\partial^2 C}{\partial K^2} \]
where we have substituted \(\frac{\partial^2 C}{\partial K^2}\) for \(P(t, T)E[\delta(S_T - K)]\). We obtain the final result, Equation (2)
\[ \frac{\partial C}{\partial T} = -K(r_T - q_T) \frac{\partial C}{\partial K} - q_T C + \frac{1}{2} K^2 E \left[ \sigma_T^2 | S_T = K \right] \frac{\partial^2 C}{\partial K^2} . \]
When \(r_T = q_T = 0\) we can re-arrange the result to obtain
\[ E \left[ \sigma_T^2 | S_T = K \right] = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}} \]
which, again, is Equation (5). Hence when the dividend and interest rate are both zero, the derivation of local volatility using Dupire’s approach and the derivation using conditional expectation produce the same result.

4 Derivation of Local Volatility From Implied Volatility, Equation (3)

To express local volatility in terms of implied volatility, we need the three derivatives \(\frac{\partial C}{\partial T}, \frac{\partial C}{\partial K}, \frac{\partial^2 C}{\partial K^2}\) that appear in Equation (1), but expressed in terms of
implied volatility. Following Gatheral [1] we define the log-moneyness

\[ y = \ln \frac{K}{F_T} \]

where \( F_T = S_0 \exp \left( \int_0^T \mu_t dt \right) \) is the forward price (\( \mu_t = r_t - q_t \), risk free rate minus dividend yield) and \( K \) is the strike price, and the "total" Black-Scholes implied variance

\[ w = \Sigma(K, T)^2 T \]

where \( \Sigma(K, T) \) is the implied volatility. The Black-Scholes call price can then be written as

\[ C_{BS} (S_0, K, \Sigma(K, T), T) = F_T \{ N(d_1) - e^y N(d_2) \} \]  

where

\[ d_1 = \frac{\ln S_0 + \int_0^T (r_t - q_t) dt + \frac{w}{2}}{\sqrt{w}} = -yw^{-\frac{1}{2}} + 1 + \frac{1}{2} w^{\frac{1}{2}} \]  

and \( d_2 = d_1 - \sqrt{w} = -yw^{-\frac{1}{2}} - \frac{1}{2} w^{\frac{1}{2}} \).

4.1 The Reparameterized Local Volatility Function

To express the local volatility Equation (1) in terms of \( y \), we note that the market call price is

\[ C(S_0, K, T) = C(S_0, F_T e^y, T) \]

and we take derivatives. The first derivative we need is, by the chain rule

\[ \frac{\partial C}{\partial y} = \frac{\partial C}{\partial K} \frac{\partial K}{\partial y} = \frac{\partial C}{\partial K} K. \]  

The second derivative we need is

\[ \frac{\partial^2 C}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial C}{\partial K} \right) K + \frac{\partial C}{\partial K} \frac{\partial K}{\partial y} \]

\[ = \frac{\partial^2 C}{\partial K^2} K^2 + \frac{\partial C}{\partial y}, \]

since by the chain rule \( \frac{\partial A}{\partial y} = \frac{\partial A}{\partial K} \frac{\partial K}{\partial y} \), so that \( \frac{\partial}{\partial y} \left( \frac{\partial C}{\partial K} \right) \frac{\partial K}{\partial y} = \frac{\partial^2 C}{\partial K^2} \frac{\partial K}{\partial y} = \frac{\partial^2 C}{\partial K^2} K \). The third derivative we need is

\[ \frac{\partial C}{\partial T} = \frac{\partial C}{\partial T} + \frac{\partial C}{\partial K} \frac{\partial K}{\partial T} \]

\[ = \frac{\partial C}{\partial T} + \frac{\partial C}{\partial K} K \mu_T \]

\[ = \frac{\partial C}{\partial T} + \frac{\partial C}{\partial y} \mu_T \]
since \( K = S_0 \exp \left( \int_0^T \mu_t \, dt + y \right) \) so that \( \frac{\partial K}{\partial T} = K \mu_T \). Equation (28) implies that

\[
\frac{\partial^2 C}{\partial K^2} K^2 = \frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y}.
\]

Now we substitute into Equation (1), reproduced here for convenience

\[
\frac{\partial C}{\partial T} = 1 - 2K^2 \frac{\partial^2 C}{\partial K^2} + \mu_T \left( C - K \frac{\partial C}{\partial K} \right) - \frac{\partial C}{\partial y} \mu_T = 1 - 2\sigma^2 \left( \frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y} \right) + \mu_T \left( C - \frac{\partial C}{\partial y} \right)
\]

which simplifies to

\[
\frac{\partial C}{\partial T} = \frac{v_L}{2} \left[ \frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y} \right] + \mu_T C \tag{30}
\]

where \( v_L = \sigma^2(K, T) \) is the local variance. This is Equation (1.8) of Gatheral [1].

### 4.2 Three Useful Identities

Before expression the local volatility Equation (1) in terms of implied volatility, we first derive three identities used by Gatheral [1] that help in this regard. We use the fact that the derivatives of the standard normal cdf and pdf are, using the chain rule, \( N'(x) = n(x)x' \) and \( n'(x) = -x n(x)x' \). We also use the relation

\[
n(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} (d_2 + \sqrt{w})^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} (d_2 + 2d_2 \sqrt{\pi + w})} = n(d_2) e^{-d_2 \sqrt{\pi - \frac{1}{2} w}} = n(d_2) e^w.
\]

From Equation (25) the first derivative with respect to \( w \) is

\[
\frac{\partial C_{BS}}{\partial w} = F_T \left[ n(d_1) d_1 - e^y n(d_2) d_{2w} \right] = F_T \left[ n(d_2) e^y \left( d_{2w} + \frac{1}{2} w^{-\frac{1}{2}} \right) - e^y n(d_2) d_{2w} \right] = \frac{1}{2} F_T e^y \left[ n(d_2) w^{-\frac{1}{2}} \right]
\]
where \( d_{1w} \) is the first derivative of \( d_1 \) with respect to \( w \) and similarly for \( d_2 \). The second derivative is

\[
\frac{\partial^2 C_{BS}}{\partial w^2} = \frac{1}{2} F_T e^y \left[ -n(d_2)d_2d_{2w}w^{-\frac{1}{2}} - \frac{1}{2}n(d_2)w^{-\frac{1}{2}} \right] \tag{31}
\]

\[
= \frac{1}{2} F_T e^y n(d_2)w^{-\frac{1}{2}} \left[ -d_2d_{2w} - \frac{1}{2}w^{-1} \right]
\]

\[
= \frac{\partial C_{BS}}{\partial w} \left[ \left( yw^{-\frac{1}{2}} + \frac{1}{2}w^{\frac{1}{2}} \right) \left( \frac{1}{2}yw^{-\frac{1}{2}} - \frac{1}{4}w^{-\frac{1}{2}} \right) - \frac{1}{2}w^{-1} \right]
\]

\[
= \frac{\partial C_{BS}}{\partial w} \left[ -\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2} \right].
\]

This is the first identity we need. The second identity we need is

\[
\frac{\partial^2 C_{BS}}{\partial w \partial y} = \frac{1}{2} F_T w^{-\frac{1}{2}} \frac{\partial}{\partial y} \left[ e^y n(d_2) \right] \tag{32}
\]

\[
= \frac{1}{2} F_T w^{-\frac{1}{2}} \left[ e^y n(d_2) - e^y n(d_2)d_2d_{2y} \right]
\]

\[
= \frac{\partial C_{BS}}{\partial w} \left[ 1 - d_2d_{2y} \right]
\]

\[
= \frac{\partial C_{BS}}{\partial w} \left( \frac{1}{2} - \frac{y}{w} \right)
\]

where \( d_{2y} = -w^{-\frac{1}{2}} \) is the first derivative of \( d_2 \) with respect to \( y \). To obtain the third identity, consider the derivative

\[
\frac{\partial C_{BS}}{\partial y} = F_T \left[ n(d_1)d_{1y} - e^y N(d_2) - e^y n(d_2)d_{2y} \right]
\]

\[
= F_T e^y \left[ n(d_2)d_{1y} - N(d_2) - n(d_2)d_{2y} \right]
\]

\[
= -F_T e^y N(d_2).
\]

The third identity we need is

\[
\frac{\partial^2 C_{BS}}{\partial y^2} = -F_T \left[ e^y N(d_2) + e^y n(d_2)d_{2y} \right] \tag{33}
\]

\[
= -F_T e^y N(d_2) + F_T e^y n(d_2)w^{-\frac{1}{2}}
\]

\[
= \frac{\partial C_{BS}}{\partial y} + 2\frac{\partial C_{BS}}{\partial w}.
\]

We are now ready for the main derivation of this section.

### 4.3 Local Volatility in Terms of Implied Volatility

We note that when the market price \( C(S_0, K, T) \) is equal to the Black-Scholes price with the implied volatility \( \Sigma(K, T) \) as the input to volatility

\[
C(S_0, K, T) = C_{BS}(S_0, K, \Sigma(K, T), T). \tag{34}
\]
We can also reparameterize the Black-Scholes price in terms of the total implied volatility \( w = \Sigma(K, T)^2 T \) and \( K = F_T e^y \). Since \( w \) depends on \( K \) and \( K \) depends on \( y \), we have that \( w = w(y) \) and we can write

\[
C(S_0, K, T) = C_{BS}(S_0, F_T e^y, w(y), T). \tag{35}
\]

We need derivatives of the market call price \( C(S_0, K, T) \) in terms of the Black-Scholes call price \( C_{BS}(S_0, F_T e^y, w(y), T) \). From Equation (35), the first derivative we need is

\[
\frac{\partial C}{\partial y} = \frac{\partial C_{BS}}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial y} \tag{36}
\]

It is easier to visualize the second derivative we need, \( \frac{\partial^2 C}{\partial y^2} \), when we express the partials in \( \frac{\partial C}{\partial y} \) as \( a, b, \) and \( c \):

\[
\frac{\partial^2 C}{\partial y^2} = \frac{\partial a}{\partial y} + \frac{\partial a}{\partial w} \frac{\partial w}{\partial y} + b(w, y) \frac{\partial c}{\partial y} + \left[ \frac{\partial b}{\partial y} + \frac{\partial b}{\partial w} \frac{\partial w}{\partial y} \right] c(y) \tag{37}
\]

\[
= \frac{\partial^2 C_{BS}}{\partial y^2} + \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{\partial w}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial y^2} + \left[ \frac{\partial^2 C_{BS}}{\partial w \partial y} + \frac{\partial^2 C_{BS}}{\partial w^2} \right] \frac{\partial w}{\partial y}
\]

\[
= \frac{\partial^2 C_{BS}}{\partial y^2} + 2 \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{\partial w}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 C_{BS}}{\partial w^2} \left( \frac{\partial w}{\partial y} \right)^2.
\]

The third derivative we need is

\[
\frac{\partial C}{\partial T} = \frac{\partial C_{BS}}{\partial T} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial T} \tag{38}
\]

\[
= \mu_T C + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial T}.
\]

Gatheral explains that the second equality follows because the only explicit dependence of \( C_{BS} \) on \( T \) is through the forward price \( F_T \), even though \( C_{BS} \) depends implicitly on \( T \) through \( y \) and \( w \). The reparameterized Dupire equation (30) is reproduced here for convenience

\[
\frac{\partial C}{\partial T} = \frac{\nu L}{2} \left[ \frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y} \right] + \mu_T C.
\]

We substitute for \( \frac{\partial C}{\partial T}, \frac{\partial^2 C}{\partial y^2}, \) and \( \frac{\partial C}{\partial y} \) from Equations (38), (37), and (36) respectively and cancel \( \mu_T C \) from both sides to obtain

\[
\frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial T} = \frac{\nu L}{2} \left[ \frac{\partial^2 C_{BS}}{\partial y^2} + 2 \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{\partial w}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 C_{BS}}{\partial w^2} \left( \frac{\partial w}{\partial y} \right)^2 \right.
\]

\[
\left. \frac{\partial C_{BS}}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial y} \right]. \tag{39}
\]
Now substitute for \( \frac{\partial^2 C_{BS}}{\partial w \partial y} \), \( \frac{\partial^2 C_{BS}}{\partial w^2} \), and \( \frac{\partial^2 C_{BS}}{\partial y^2} \) from the identities in Equations (31), (32), and (33) respectively, the idea being to end up with terms involving \( \frac{\partial C_{BS}}{\partial w} \) on the right hand side of Equation (39) that can be factored out.

\[
\frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial T} = \frac{v_L}{2} \frac{\partial C_{BS}}{\partial w} \left[ 2 + 2 \left( \frac{1}{2} - \frac{y}{w} \right) \frac{\partial w}{\partial y} + \left( -\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w} \right) \left( \frac{\partial w}{\partial y} \right)^2 \right. \\
\left. + \frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial y} \right].
\]

Remove the factor \( \frac{\partial C_{BS}}{\partial w} \) from both sides and simplify to obtain

\[
\frac{\partial w}{\partial T} = v_L \left[ 1 - \frac{y}{w} \frac{\partial w}{\partial y} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2} + \frac{1}{4} \left( -\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w} \right) \left( \frac{\partial w}{\partial y} \right)^2 \right].
\]

Solve for \( v_L \) to obtain the final expression for the local volatility expressed in terms of implied volatility \( w = \Sigma (K, T)^2 T \) and the log-moneyness \( y = \ln \frac{K}{F_T} \).

\[
v_L = \frac{\frac{\partial w}{\partial T}}{\left[ 1 - \frac{y}{w} \frac{\partial w}{\partial y} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2} + \frac{1}{4} \left( -\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w} \right) \left( \frac{\partial w}{\partial y} \right)^2 \right]}.\]

### 4.4 Alternate Derivation

In this derivation we express the derivatives \( \frac{\partial C}{\partial K} \), \( \frac{\partial^2 C}{\partial K \partial y} \), and \( \frac{\partial^2 C}{\partial y^2} \) in the Dupire equation (1) in terms of \( y \) and \( w = w(y) \), but we substitute these derivatives directly in Equation (1) rather than in (30). This means that we take derivatives with respect to \( K \) and \( T \), rather than with \( y \) and \( T \). Recall that from Equation (35), the market call price is equal to the Black-Scholes call price with implied volatility as input

\[
C(S_0, K, T) = C_{BS}(S_0, F_T e^y, w(y), T).
\]

Recall also that from Equation (25) the Black-Scholes call price reparameterized in terms of \( y \) and \( w \) is

\[
C_{BS} (S_0, F_T e^y, w(y), T) = F_T \{ N(d_1) - e^y N(d_2) \}
\]

where \( d_1 \) is given in Equation (26), and where \( d_2 = d_1 - \sqrt{w} \). The first derivative we need is

\[
\frac{\partial C}{\partial K} = \frac{\partial C_{BS}}{\partial y} \frac{\partial y}{\partial K} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial K} = \frac{1}{K} \frac{\partial C_{BS}}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial K}.
\]
The second derivative is
\[
\frac{\partial^2 C}{\partial K^2} = -\frac{1}{K^2} \frac{\partial C_{BS}}{\partial y} + \frac{1}{K} \frac{\partial}{\partial K} \left( \frac{\partial C_{BS}}{\partial y} \right),
\]
\[
+ \frac{\partial}{\partial K} \left( \frac{\partial C_{BS}}{\partial w} \right) \frac{\partial w}{\partial K} + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial K^2},
\]
\[\text{(41)}\]

Let \( A = \frac{\partial C}{\partial y} \) for notational convenience. Then \( \frac{\partial}{\partial K} \left( \frac{\partial C}{\partial y} \right) = \frac{\partial A}{\partial K} \) and
\[
\frac{\partial}{\partial K} \left( \frac{\partial C_{BS}}{\partial y} \right) = \frac{\partial A}{\partial K} \frac{\partial y}{\partial K} + \frac{\partial A}{\partial w} \frac{\partial w}{\partial K} = \frac{\partial^2 C_{BS}}{\partial y^2} \frac{1}{K} + \frac{\partial^2 C_{BS}}{\partial w^2} \frac{\partial w}{\partial K},
\]
\[\text{(42)}\]

Similarly
\[
\frac{\partial}{\partial K} \left( \frac{\partial C_{BS}}{\partial w} \right) = \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{1}{K} + \frac{\partial^2 C_{BS}}{\partial w^2} \frac{\partial w}{\partial K}.
\]
\[\text{(43)}\]

Substituting Equations (42) and (43) into Equation (41) produces
\[
\frac{\partial^2 C}{\partial K^2} = -\frac{1}{K^2} \frac{\partial C_{BS}}{\partial y} + \frac{1}{K} \left( \frac{\partial^2 C_{BS}}{\partial y^2} \frac{1}{K} + \frac{\partial^2 C_{BS}}{\partial w^2} \frac{\partial w}{\partial K} \right)
\]
\[
+ \left( \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{1}{K} \frac{\partial w}{\partial K} + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial K^2} \right) = \frac{1}{K^2} \left( \frac{\partial^2 C_{BS}}{\partial y^2} - \frac{\partial C_{BS}}{\partial y} \right) + \frac{2}{K} \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{\partial w}{\partial K}
\]
\[
+ \frac{\partial^2 C_{BS}}{\partial w^2} \left( \frac{\partial w}{\partial K} \right)^2 + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial K^2}.
\]
\[\text{(44)}\]

The third derivative we need is
\[
\frac{\partial C}{\partial T} = \frac{\partial C_{BS}}{\partial T} + \frac{\partial C_{BS}}{\partial y} \frac{\partial y}{\partial T} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial T}
\]
\[\text{(45)}\]
\[
= \mu_T C_{BS} + \frac{\partial C_{BS}}{\partial y} \mu_T + \frac{\partial C_{BS}}{\partial w} \mu_T,
\]

again using the fact that \( \frac{\partial C_{BS}}{\partial y} \) depends explicitly on \( T \) only through \( F_T \). Now substitute for \( \frac{\partial C}{\partial K}, \frac{\partial C}{\partial y}, \) and \( \frac{\partial C}{\partial T} \) from Equations (40), (44), and (45) respectively into Equation (4) for Dupire local variance, reproduced here for convenience.
\[
\sigma^2 = \frac{\frac{\partial C}{\partial T} - \mu_T [C_{BS} - K \frac{\partial C}{\partial K}]}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}.
\]

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We obtain, after applying the three useful identities in Section 4.2,

\[
\sigma^2 = \mu_T C_{BS} + \frac{\partial C_{BS}}{\partial y} \mu_T + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial T} - \mu_T \left( C_{BS} - K \left( \frac{1}{K} \frac{\partial C_{BS}}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial K} \right) \right) + \frac{1}{2} K^2 \left( \frac{1}{K} \left( \frac{\partial^2 C_{BS}}{\partial y^2} - \frac{\partial C_{BS}}{\partial y} \right) \right) + 2 \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{\partial w}{\partial K} + \frac{\partial^2 C_{BS}}{\partial w^2} \left( \frac{\partial w}{\partial K} \right)^2 + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial T \partial K}.
\]

Applying the three useful identities in Section 4.2 allows the term \(\frac{\partial C_{BS}}{\partial w}\) to be factored out of the numerator and denominator. The last equation becomes

\[
\sigma^2 = \frac{1}{2} K^2 \left[ \frac{1}{K} \left( \frac{\partial^2 C_{BS}}{\partial y^2} - \frac{\partial C_{BS}}{\partial y} \right) \right] + 2 \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{\partial w}{\partial K} + \frac{\partial^2 C_{BS}}{\partial w^2} \left( \frac{\partial w}{\partial K} \right)^2 + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial T \partial K}.
\]

Equation (46) can be simplified by considering deriving the partial derivatives of the Black-Scholes total implied variance, \(w = \Sigma(K, T) T\). We have \(\frac{\partial w}{\partial T} = 2 \Sigma T \frac{\partial \Sigma^2}{\partial T} + \Sigma^2, \frac{\partial w}{\partial K} = 2 \Sigma T \frac{\partial \Sigma}{\partial K}\), and \(\frac{\partial^2 w}{\partial K^2} = 2 T \left( \frac{\partial \Sigma^2}{\partial K} \right)^2 + \Sigma \frac{\partial^2 \Sigma}{\partial K^2}\). Substitute into Equation (46). The numerator in Equation (46) becomes

\[
\Sigma^2 + 2 \Sigma T \left( \frac{\partial \Sigma}{\partial T} + \mu_T K \frac{\partial \Sigma}{\partial K} \right)
\]

and the denominator becomes

\[
1 + 2 K \Sigma T \left( \frac{1}{2} - \frac{y}{w} \right) \frac{\partial \Sigma}{\partial K} + 2 K^2 \Sigma^2 T^2 \left( -\frac{1}{8} - \frac{1}{2 \Sigma^2 T} + \frac{y^2}{2 w^2} \right) \left( \frac{\partial \Sigma}{\partial K} \right)^2 + K^2 T \left[ \left( \frac{\partial \Sigma}{\partial K} \right)^2 + \Sigma \frac{\partial^2 \Sigma}{\partial K^2} \right].
\]

Replacing \(w\) with \(\Sigma^2 T\) everywhere in the denominator produces

\[
1 + 2 K \Sigma T \left( \frac{1}{2} - \frac{y}{\Sigma^2 T} \right) \frac{\partial \Sigma}{\partial K} + 2 K^2 \Sigma^2 T^2 \left( -\frac{1}{8} - \frac{1}{2 \Sigma^2 T} + \frac{y^2}{2 \Sigma^2 T^2} \right) \left( \frac{\partial \Sigma}{\partial K} \right)^2 + K^2 T \left[ \left( \frac{\partial \Sigma}{\partial K} \right)^2 + \Sigma \frac{\partial^2 \Sigma}{\partial K^2} \right] = 1 + K \Sigma T \frac{\partial \Sigma}{\partial K} - \frac{2 K y \partial \Sigma}{\Sigma} \frac{\partial \Sigma}{\partial K} - \frac{K^2 \Sigma^2 T^2}{4} \left( \frac{\partial \Sigma}{\partial K} \right)^2 + \frac{K^2 y^2}{\Sigma^2} \left( \frac{\partial \Sigma}{\partial K} \right)^2 + K^2 \Sigma T \frac{\partial^2 \Sigma}{\partial K^2}
\]

\[
= \left( 1 - \frac{K y \partial \Sigma}{\Sigma} \frac{\partial \Sigma}{\partial K} \right)^2 + \left[ 1 - 2 K y \frac{\partial \Sigma}{\Sigma} \frac{\partial \Sigma}{\partial K} + \left( \frac{K y \partial \Sigma}{\Sigma} \right)^2 \right].
\]

Substituting the numerator in (47) and the denominator in (48) back to Equation (46), we obtain

\[
\frac{\Sigma^2 + 2 \Sigma T \left( \frac{\partial \Sigma}{\partial T} + \mu_T K \frac{\partial \Sigma}{\partial K} \right)}{\left( 1 + \frac{K y \partial \Sigma}{\Sigma} \frac{\partial \Sigma}{\partial K} \right)^2 + K \Sigma T \left[ \frac{\partial \Sigma}{\partial K} - \frac{1}{4} K \Sigma T \left( \frac{\partial \Sigma}{\partial K} \right)^2 + K \frac{\partial^2 \Sigma}{\partial K^2} \right]}
\]
See also the dissertation by van der Kamp [4] for additional details of this alternate derivation.

References


