Heuristic Derivation of the Fokker-Planck Equation

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1 The SDE and its Transition Density

Start with the SDE defined by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.$$

The transition density $\rho(x,t|y,s)$ is defined by

$$\begin{aligned}\int_A \rho(x,t|y,s)dx &= \Pr\left[X_{t+s} \in A | X_s = y\right] \\ &= \Pr\left[X_t \in A | X_0 = y\right].\end{aligned}$$

The density $\rho(x,t|y,s)$ is time-invariant since $\mu(X_t)$ and $\sigma(X_t)$ are assumed to be time invariant, and consequently, that X_t is assumed to be stationary.

2 Derivation of the Equation

Consider a differentiable function $V(X_t,t) = V(x,t)$ with $V(X_t,t) = 0$ for $t \notin (0,T)$. Then by Itō's Lemma

$$dV = \left[\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}\right]dt + \left[\sigma\frac{\partial V}{\partial x}\right]dW_t$$

so that

$$V(X_T,T) - V(X_0,0) = \int_0^T \left[\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}\right]dt + \int_0^T \left[\sigma\frac{\partial V}{\partial x}\right]dW_t \quad (1)$$

where $\mu = \mu(X_t)$ and $\sigma = \sigma(X_t)$ for notational convenience. Take the conditional expectation of both sides of equation (1) given X_0

$$\begin{aligned}&E\left[V(X_T,T) - V(X_0,0)\right] \qquad (2)\\ =\ & E\int_0^T \left[\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}\right]dt + E\int_0^T \left[\sigma\frac{\partial V}{\partial x}\right]dW_t \\ =\ & \int_{\mathbb{R}} \left\{\int_0^T \left[\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}\right]dt\right\}\rho(x,t|y,s)dx.\end{aligned}$$

In this note, all expectations are expectations conditional on X_0, so that $E[\cdot] = E[\cdot|X_0 = y]$. Since $E\left[dW_t\right] = 0$, the second term in the middle line of equation (2) drops out. Hence, we can write equation (2) as three integrals

$$\int_{\mathbb{R}}\int_0^T \rho\frac{\partial V}{\partial t}dtdx + \int_{\mathbb{R}}\int_0^T \rho\mu\frac{\partial V}{\partial x}dtdx + \frac{1}{2}\int_{\mathbb{R}}\int_0^T \rho\sigma^2\frac{\partial^2 V}{\partial x^2}dtdx = I_1 + I_2 + I_3$$

where $\rho = \rho(x,t|y,s)$ for notational convenience. The objective of the derivation is to apply integration by parts to get rid of the derivatives of V.

2.1 Evaluation of the Integrals

The trick is that I_1 is evaluated using integration by parts on t, while I_2 and I_3 are each evaluated using integration by parts on x.

2.1.1 Evaluation of I_1

Use $u = \rho, v' = \frac{\partial V}{\partial t}$ so that $u' = \frac{\partial \rho}{\partial t}$ and $v = V$. Hence for the inside integrand of I_1 we have

$$\int_0^T \rho \frac{\partial V}{\partial t} dt = \rho V|_0^T - \int_0^T \frac{\partial \rho}{\partial t} V dt = -\int_0^T \frac{\partial \rho}{\partial t} V dt$$

since at the boundaries 0 and T, $V = 0$. Hence

$$I_1 = -\int_{\mathbb{R}} \int_0^T \frac{\partial \rho}{\partial t} V(x,t) dt dx. \tag{3}$$

2.1.2 Evaluation of I_2

Change the order of integration in I_2 and write it as

$$I_2 = \int_0^T \int_{\mathbb{R}} \rho\mu \frac{\partial V}{\partial x} dx dt.$$

Use integration by parts on the integrand, with $u = \rho\mu, v' = \frac{\partial V}{\partial x}$ so that $u' = \frac{\partial(\rho\mu)}{\partial x}, v = V$

$$\int_{\mathbb{R}} \rho\mu \frac{\partial V}{\partial x} dx = \rho\mu V|_{\mathbb{R}} - \int_{\mathbb{R}} \frac{\partial(\rho\mu)}{\partial x} V dx.$$

Hence the integral can be evaluated as

$$\begin{aligned} I_2 &= -\int_0^T \int_{\mathbb{R}} \frac{\partial(\rho\mu)}{\partial x} V(x,t) dx dt \\ &= -\int_{\mathbb{R}} \int_0^T \frac{\partial(\rho\mu)}{\partial x} V(x,t) dt dx. \end{aligned} \tag{4}$$

2.1.3 Evaluation of I_3

Finally, the evaluation of the integrand of I_3 requires the application of integration by parts on x twice. This is because in the integrand we want to get rid of the $\frac{\partial^2 V}{\partial x^2}$ term and end up with $V(x,t)$ only. Again, change the order of integration and write I_3 as

$$\frac{1}{2} \int_0^T \int_{\mathbb{R}} \rho\sigma^2 \frac{\partial^2 V}{\partial x^2} dx dt.$$

For the first integration by parts use $u = \rho\sigma^2, v' = \frac{\partial^2 V}{\partial x^2}$ so that $u' = \frac{\partial(\rho\sigma^2)}{\partial x}$ and $v = \frac{\partial V}{\partial x}$. Hence the integrand can be written

$$\begin{aligned}\int_{\mathbb{R}} \rho\sigma^2 \frac{\partial^2 V}{\partial x^2} dx &= \rho\sigma^2 \frac{\partial V}{\partial x}\bigg|_{\mathbb{R}} - \int_{\mathbb{R}} \frac{\partial(\rho\sigma^2)}{\partial x}\frac{\partial V}{\partial x} dx \\ &= -\int_{\mathbb{R}} \frac{\partial(\rho\sigma^2)}{\partial x}\frac{\partial V}{\partial x} dx.\end{aligned}$$

Apply integration by parts again, with $u = \frac{\partial(\rho\sigma^2)}{\partial x}, v' = \frac{\partial V}{\partial x}, u' = \frac{\partial^2(\rho\sigma^2)}{\partial x^2}, v = V$

$$\begin{aligned}-\int_{\mathbb{R}} \frac{\partial(\rho\sigma^2)}{\partial x}\frac{\partial V}{\partial x} dx &= -\frac{\partial(\rho\sigma^2)}{\partial x} V\bigg|_{\mathbb{R}} + \int_{\mathbb{R}} \frac{\partial^2(\rho\sigma^2)}{\partial x^2} V dx \\ &= \int_{\mathbb{R}} \frac{\partial^2(\rho\sigma^2)}{\partial x^2} V(x,t) dx.\end{aligned}$$

This implies that I_3 can be written as

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}} \frac{\partial^2(\rho\sigma^2)}{\partial x^2} V dx dt = \frac{1}{2}\int_{\mathbb{R}} \int_0^T \frac{\partial^2(\rho\sigma^2)}{\partial x^2} V(x,t) dt dx. \tag{5}$$

2.1.4 Obtaining the Equation

Substitute equations (3), (4), and (5) into equation (2)

$$\begin{aligned} & E\left[V(X_T,T)\right] - V(X_0,0) \\ = & -\int_{\mathbb{R}}\int_0^T \frac{\partial \rho}{\partial t} V(x,t) dt dx - \int_{\mathbb{R}}\int_0^T \frac{\partial(\rho\mu)}{\partial x} V(x,t) dt dx \\ & +\frac{1}{2}\int_{\mathbb{R}}\int_0^T \frac{\partial^2(\rho\sigma^2)}{\partial x^2} V(x,t) dt dx \\ = & \int_{\mathbb{R}}\int_0^T V(x,t)\left[-\frac{\partial \rho}{\partial t} - \frac{\partial(\rho\mu)}{\partial x} + \frac{1}{2}\frac{\partial^2(\rho\sigma^2)}{\partial x^2}\right] dt dx.\end{aligned}$$

Since $V(X_t,t) = 0$ for $t \notin (0,T)$ we have $V(X_T,T) = V(X_0,0) = 0$ so that $E\left[V(X_T,T)\right] - V(X_0) = 0$. This implies that the portion of the integrand in the brackets is zero

$$-\frac{\partial \rho}{\partial t} - \frac{\partial(\rho\mu)}{\partial x} + \frac{1}{2}\frac{\partial^2(\rho\sigma^2)}{\partial x^2} = 0$$

from which the Fokker-Planck equation can be obtained

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho\mu)}{\partial x} + \frac{1}{2}\frac{\partial^2(\rho\sigma^2)}{\partial x^2}.$$